Vibrations of a Stiffened Capped Cylinder

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Theme

THE free vibrations of a shell composed of a ring-reinforced cylinder with spherical end caps (Fig. 1) are calculated theoretically. First, the eigenvectors of the unreinforced capped shell are obtained. These are then used as expansion modes for the reinforced shell, thereby resulting in an infinite set of coupled simultaneous equations. The final eigenvalues and eigenvectors are then determined by solving the truncated set. The results point out the influence of the ring stiffness and spacing on the frequencies and mode shapes. Comparison is also made with results obtained for unreinforced cylindrical shells having various edge constraints.

Content

An extensive literature exists on the free vibrations of shell elements. The preponderance of those studies have considered unreinforced and reinforced cylindrical and shallow and non-shallow spherical shells. Few vibration studies of structures composed of combinations of simple shell elements have been reported on. However, one of these is directly related to the present investigation. Harari and Baron[1] determined the free vibrational response of a shell consisting of a reinforced cylinder with spherical end caps. Each of the shell elements and stiffeners were treated as discrete members, and compatibility was enforced at their junctures. The present study determines the free vibrations of a similar shell configuration by using the unreinforced shell eigenvectors as expansion modes. The introduction of reinforcing members results in coupling of these modes.

It can be readily shown[2] that for a spherical shell the free vibrational equations of motion based on Sanders[3] shell theory can be expressed as the following set of uncoupled equations.

$$(\nabla^2 + r_1)(\nabla^2 + r_2)(\nabla^2 + r_3)W = 0;$$
$$\nabla^2\psi + 2\psi = -[2/(1+k)(1-\nu)]\omega^2\psi$$

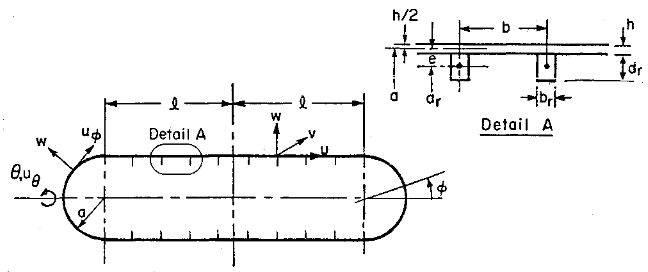

Fig. 1 Geometry of capped shell.

and

$$a_1 a_2 U = a_3 \nabla^4 W + [2a_3 + ka_2]\nabla^2 W + [a_4 - a_2 a_5] W + a_6 a_2 \sin\varphi\psi,_\varphi \qquad (1)$$

The nondimensional displacements u_φ, u_θ are defined by the auxiliary functions U, ψ so that

$$u_\varphi = [U,_\varphi - \psi\sin\varphi]e^{-i\omega\tau};\quad u_\theta = U,_\theta \csc\varphi e^{-i\omega\tau};\quad w = We^{-i\omega\tau} \qquad (2)$$

Nondimensional time $\tau = c_s t/a$; the shell velocity $c_s = [E/\rho(1-\nu^2)]^{1/2}$; the shell parameter $k^2 = h^2/12a^2$ and

$$a_1 = (1+k)(1-\nu) + \omega^2;\quad a_2 = (1+k)(1+\nu) + k\omega^2;$$
$$a_3 = k(1+k);\quad a_4 = (1+k)^2[2(1+\nu) - \omega^2];$$
$$a_5 = (1+k)(1+\nu) - 2k;\quad a_6 = -(1+k)(1-\nu)/2 \qquad (3)$$

The r_i's of Eq. (1) are the roots of the cubic

$$r^3 - C_1 r^2 + C_2 r - C_3 = 0 \qquad (4)$$

where

$$C_1 = 4 + \omega^2;\quad C_2 = (5 - \nu^2 - \nu\omega^2) + (1 - \nu^2 - \omega^2)/k;$$
$$C_3 = 2(1-\nu^2) - (1-\nu)\omega^2 + [2(1-\nu^2) + (1+3\nu)\omega^2 - \omega^4] \qquad (5)$$

The solution of Eqs. (1a, b) is

$$W = \sum_{m=0}^{\infty}\sum_{\alpha=1}^{3} A_\alpha^m P_{\mu_\alpha}^m(\cos\varphi)\cos m\theta;$$
$$\psi = \sum_{m=0}^{\infty} C^m P_\eta^m(\cos\varphi)\cos m\theta \qquad (6)$$

where $P_{\mu_\alpha}^m$, P_η^m are the associated Legendre functions of the first kind, and

$$\mu_\alpha = [(1/4) + r_\alpha]^{1/2} - 1/2;\quad \eta = [(1/4) + 2[1 + \omega^2/(1+k)(1-\nu)]^{1/2} - 1/2 \qquad (7)$$

Substituting Eqs. (6) into Eq. (1c) then yields

$$a_1 a_2 U = \sum_{m=0}^{\infty}\sum_{\alpha=1}^{3}\{A_\alpha^m[a_3 r_\alpha^2 - (2a_3 + ka_2)r_\alpha + (a_4 - a_2 a_5)] \times P_{\mu_\alpha}^m(z)\cos m\theta\} - a_6 a_2 \sin^2\varphi \sum_{m=0}^{\infty} C^m P_\eta^m(z),_z \cos m\theta \qquad (8)$$

The displacements and stress resultants are now determined by substituting the expressions for W, ψ and U into the appropriate equations.

The substitution of the displacement forms

$$u = \sum_{i=1}^{8} D_i e^{\lambda_i x}\cos m\theta e^{-i\omega\tau};\quad v = \sum_{i=1}^{8} E_i e^{\lambda_i x}\sin m\theta e^{-i\omega t};$$
$$w = \sum_{i=1}^{8} F_i e^{\lambda_i x}\cos m\theta e^{-i\omega\tau} \qquad (9)$$

into Sanders' cylindrical shell equations, results in the following fourth-order equation in λ^2

$$d_4\lambda^8 + d_3\lambda^6 + d_2\lambda^4 + d_1\lambda^2 + d_0 = 0 \qquad (10)$$

Received April 2, 1975; synoptic received Oct. 22, 1974; revision received April 13, 1976. Full paper available from National Technical Information Service, Springfield, Va. 22151, as N76-20517 at the standard price (available upon request). This work was sponsored by the Office of Naval Research, Department of the Navy, under Contract No. N00014-67-A-0438-0006.

Index category: Structural Dynamic Analysis.

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where the d's are functions of the modal number m and frequency ω. For the θ-independent motion, the reduced forms of Eq. (10) are

$$\lambda^6+\omega^2\lambda^4+\frac{1}{k}(1-\nu^2-\omega^2)\lambda^2+\omega^2(1-\omega^2)=0;$$

$$\lambda^2=-\frac{2\omega^2/(1-\nu)}{[1+(9/4)k]} \tag{11}$$

At the spherical-cylindrical shell interfaces the displacements, slopes, and stress resultants must be compatible. This provides a system of 16 (12, when $m=0$) homogeneous linear algebraic equations, the solution of which yields the frequencies and corresponding eigenvectors.

Upon expressing the motion of the reinforced shell in terms of a free vibrational modal expansion, the interaction ring forces appear as external forces acting on the shell surface. Assuming an harmonic solution for the generalized coordinates

$$q_{mn}=q_{mn}e^{-i\Omega\tau} \tag{12}$$

we obtain the following system of simultaneous homogeneous equations on the modal amplitudes

$$q_{mn}=\sum_j Q^j_{mn}/M_{mn}(\omega^2_{mn}-\Omega^2) \tag{13}$$

where ω_{mn} = eigenvalues of unstiffened shell, Ω = stiffened shell frequency, M_{mn} = generalized mass component; the j^{th} ring generalized interaction force

$$Q^j_{mn}=\frac{2\pi}{\epsilon_m}\left[F^j_u U_{mn}(x^j)+F^j_v V_{mn}(x^j)+F^j_w W_{mn}(x^j)+\frac{F^j_\psi}{a}\frac{dW_{mn}(x^j)}{dx^j}\right] \tag{14}$$

where $\epsilon_m = 1\ (m=0),\ 2\ (m>0)$;

$$F^j_u=-k_{xx}\sum_{n=1}^{\infty}q_{mn}U_{mn}(j)-k_{x\psi}\sum_{n=1}^{\infty}q_{mn}W_{mn,x}(j)$$

$$F^j_v=-\left(1-\frac{e}{a}\right)\left[k_{r\theta}\sum_{n=1}^{\infty}q_{mn}W_{mn}(j)+k_{\theta\theta}\sum_{n=1}^{\infty}q_{mn}V_{mn}(j)\right]$$

$$F^j_w=-k_{rr}\sum_{n=1}^{\infty}q_{mn}W_{mn}(j)-k_{r\theta}\left[\left(1-\frac{e}{a}\right)\times\sum_{n=1}^{\infty}q_{mn}V_{mn}(j)-\frac{e}{a}m\sum_{n=1}^{\infty}q_{mn}W_{mn}(j)\right]$$

$$F^j_\psi=-ak_{x\psi}\left[\sum_{n=1}^{\infty}q_{mn}U_{mn}(j)+\frac{e}{a}\sum_{n=1}^{\infty}q_{mn}W_{mn,x}(j)\right]-k_{\psi\psi}\times\sum_{n=1}^{\infty}q_{mn}W_{mn,x}(j) \tag{15}$$

and the ring stiffness k_{ij} are functions of the extensional bending and torsional stiffnesses, density, and eccentricity of the ring, the modal number m and frequency Ω.

Calculations have been performed for a shell having radius $a=4.082$ in., thickness $h=0.047$ in., and a cylindrical segment of length $2l=18.54$ in. The material constants used are $E=30\times10^6$ lb/in^2, $\rho=0.7338\times10^{-3}$ lb sec^2/in^4, $\nu=0.3$. As a basis for comparison, frequencies have also been obtained for unreinforced cylindrical shells having the following end conditions: a) simple supports ($v=w=0$, $N_x=M_x=0$); b) clamped supports ($v=w=w_{,x}=N_x=0$); c) clamped supports ($u=v=w=w_x=0$); d) free ends ($N_x=V_x=T_{x\theta}=M_x=0$). For the reinforced capped shell, the following stiffener parameters were used: number, $N_s=5$, 14; spacing, $b=3.02$, 1.15 in.

The results for the lowest frequencies of the unreinforced shell (Fig. 2) indicate that at the higher circumferential wave numbers the frequencies are essentially unaffected by the end conditions. As has been previously pointed out[4] the strain

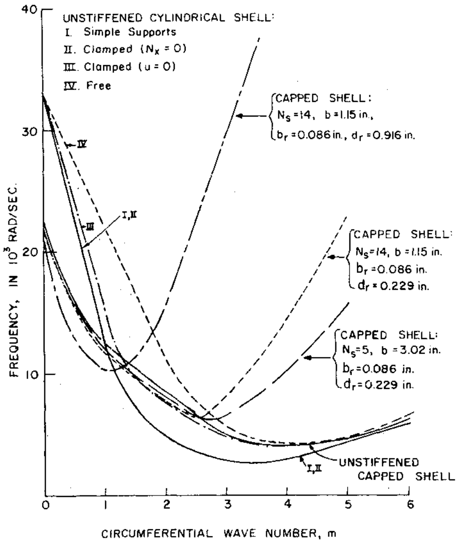

Fig. 2 Lowest frequencies.

energy of the lowest frequency is primarily extensional for the lower circumferential wave numbers, while at the higher wave numbers bending energy dominates. Thus since the stretching energy is intimately related to the end condition, only the lower wave number frequencies are affected by them. An exception of this occurs at $m=0$ where the frequencies are independent of the end conditions except for the capped shell. The reduced frequency of the capped shell can be ascribed to the large axial end displacements which results in a large increase in the kinetic energy due to the fact that the spherical caps behave essentially like rigid end masses. The free shell exhibits the highest frequencies, while the simply supported and clamped ($N_x=0$) shells, whose lowest frequencies coincide, have the lowest frequencies. At $m=1$, the capped shell frequency coincides with that of the simply supported and clamped ($N_x=0$) shells; for $m>2$ the frequency of the capped shell lies between that of the free and clamped ($u=0$) shells. The relatively low value of the cappped shell frequency at $m=1$ can be directly attributed to the essentially rigid body behavior of the spherical caps.

The frequencies for the capped shells, are similar to those obtained for the simply supported cylindrical shell by Wah and Hu.[5] Above a critical value of m, the attachment of the stiffeners increase the frequencies, but below that value, the addition of stiffiners results in reduced values of the frequency. At the higher circumferential wave numbers m the heavy rings ($d_r=0.916$in.)have in-plane flexural frequencies which are very much greater than the corresponding torsional frequencies. This accounts for the multi-nodal displacement configuration exhibited by them. The opposite situation occurs for the light rings; i.e., the torsional frequencies are much greater than the in-plane flexural frequencies. When the shell is stiffened by these closely-spaced light rings, inter-ring displacement does not occur and the reinforced shell acts like an orthopedic shell. The inter-ring displacements become more pronounced with increasing circumferential wave number m. Decreasing the number of rings N_s will, of course, also result in an increase in the inter-ring displacements.

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